

Stability of a Coastal Upwelling Front

1. Model Development and a Stability Theorem

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A two-layer shallow water equation model is used to investigate the linear stability of a coastal upwelling front. The model features a surface front parallel to a coastal boundary and bottom topography which is an arbitrary function of the cross-shelf coordinate. Global conservation statements for energy, momentum and potential vorticity are examined to help elucidate the instability mechanism. By combining these conservation statements, a general stability theorem is established which allows the a priori determination of the stability of a coastal upwelling front. The necessary conditions for instability derived for these ageostrophic flows differ from traditional quasi-geostrophic criteria. The stability theorem suggests that a coastal upwelling front may be unstable no matter what the basic state flow configuration is. Confirmation of the existence of these unstable waves, a description of their characteristics and a comparison of their properties to observations are presented in a companion paper.

1. INTRODUCTION

Fronts, regions of sharp, horizontal contrast in some fluid property (e.g., temperature, salinity or density), are a common feature in the ocean and atmosphere. They are often regions of large velocities and velocity gradients which are fundamental to the structure of the oceanic or atmospheric circulation. Near coastal barriers, several varieties of fronts can be identified. One prominent type of front in the coastal ocean is due to the process of upwelling. An alongshore wind directed so that the coastal barrier is to its left in the northern hemisphere drives an offshore Ekman flux in the upper part of the water column. This offshore flux requires some combination of horizontal and vertical flow to conserve the volume of seawater. The resulting sharp, near-surface horizontal density contrast between the less dense surface water offshore and the newly upwelled water inshore is called the coastal upwelling front.

Upwelling occurs in many regions of the world's coastal oceans, including off the coasts of western North America, southwestern and northwestern Africa, and Nova Scotia. Examination of a cross-shelf vertical section of density from one of these areas during active upwelling reveals a region of compressed density contours intersecting the surface 10–50 km offshore and continuing seaward at approximately 20–70 m depth. If alongshore winds were steady and the coastline and bottom topography uniform in the downwind direction, the coastal upwelling front would tend to parallel the coastline. Horizontal maps of surface properties (usually temperature because of its relative ease of measurement by, for example, remote sensing techniques) often indicate a great deal of alongshore variability in the offshore position of the front. An example of this alongfront variability is revealed in a map of sea surface temperature off the coast of Oregon as measured by an airborne radiome-

ter (Figure 1). The coastal upwelling front is the region of compressed isotherms approximately 20 km offshore with cold, upwelled water lying closest to shore. There exists a wavelike meander in the front with an alongfront wavelength of 30–50 km. Alongfront variability on these scales is a common feature of coastal upwelling fronts. Other observations suggest that these disturbances can extend alongshore over many repeated wavelengths [Breaker and Mooers, 1986] and that the amplitude of the meanders can grow with time [Petrie *et al.*, 1987]. While several mechanisms can be suggested to explain alongfront variability in the coastal upwelling front (for example, the influence of alongshore topographic variations), their wavelike nature and observed growth suggest that they may be unstable waves which amplify in the absence of external forcing.

Upwelling fronts are an important feature of the coastal ocean, and an understanding of their variability is essential to a complete description of coastal circulation. Fronts in density and other water properties (e.g., nutrient content) associated with the coastal upwelling front play an important role in the biology of the coastal environment. Meanders in the coastal upwelling front direct the normally alongshelf coastal flow cross-shelf and may, with time, grow to the point where they break the continuity of the alongshelf flow by forming detached eddies. Another possibility is that the meander peaks away from the coast may grow in the offshore direction, creating strong cross-shore flows well seaward of the continental shelf break (~30 km offshore on the western coast of the United States). This would serve to transport large amounts of cold, nutrient-rich upwelled water off the shelf. These so-called offshore jets or squirts have been documented in a number of studies off the western coast of the United States [e.g., Davis, 1985; Kosro, 1987]. Clearly, it would be useful to understand the formation of these frontal meanders and to be able to predict their alongfront length scale, rate of growth, and other relevant properties.

The goal of this study is to show that wavelike disturbances with the properties of observed frontal variability can be produced by hydrodynamic instability. The approach is to examine the potential for unstable disturbances to form on a basic state flow which has previously arisen from the influence of an alongshore wind via traditional upwelling pro-

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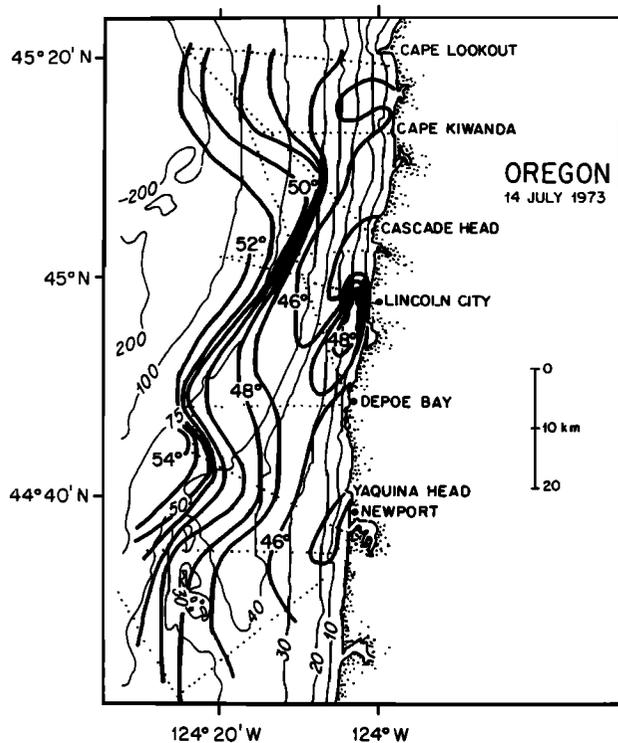


Fig. 1. Sea surface temperature map off the coast of Oregon derived from aircraft radiometer measurements taken on July 14, 1973. Temperature contours (dashed curves) are in degrees Fahrenheit and isobaths are in fathoms (1 fm = 1.83 m). Flight path is shown as a dotted line. From O'Brien et al. [1974].

cesses. The governing equations employed are the shallow water equations rather than the quasi-geostrophic equations [Pedlosky, 1986] because the latter, while simplifying the instability calculation, are inapplicable to frontal regions. Large interface displacements, strong horizontal shears, and large slopes in the bottom topography (which are allowed in this study) are not allowed in quasi-geostrophic theory. The inclusion of ageostrophic dynamics will substantially modify the well-known results of quasi-geostrophic stability theory.

Models of the formation of coastal upwelling fronts provide the basic state density and velocity fields used in the stability analysis presented here. In these models [e.g., Csanady, 1971; Pedlosky, 1978], the scale over which the interface or density surfaces warp upward to contact the sea surface (in an approximately exponential shape) is generally the internal Rossby radius of deformation (typically 5–10 km). A model which allows entrainment between layers due to wind mixing [de Szoeke and Richman, 1984] describes the offshore migration of the surface front, a feature which is observed in nature.

One of the first studies of frontal instability using the shallow water equations was that of Orlanski [1968]. He studied a two-layer Margules front intersecting flat top and bottom boundaries and explored a wide range of Rossby number-Richardson number space, finding unstable waves at all wavenumbers. Orlanski [1969] extended the model to include arbitrary interface and bottom profiles with the goal of modelling unstable waves in the Gulf Stream. The presence of sloping bottom topography was found to stabilize (decrease the growth rates of the unstable waves, but not elim-

inate them) the system which agrees with quasi-geostrophic results [e.g., Mechoso and Sinton, 1981]. Orlanski concentrated on the Gulf Stream problem and did not model a surface front over a continental shelf near a coast such as the coastal upwelling front.

Recently, progress on frontal instability has been made using one-layer reduced gravity models [e.g., Killworth and Stern, 1982; Griffiths et al. 1982; Killworth, 1983; Hayashi and Young, 1987]. These studies yield unstable modes with growth rates much smaller than those observed in laboratory experiments, which necessarily include active lower layers [e.g., Griffiths et al. 1982; Griffiths and Linden, 1982; Chia et al. 1982; Narimousa and Marworthy, 1987]. The reduced gravity models exclude the potentially powerful mechanism of true baroclinic instability. Killworth et al. [1984] studied a two-layer isolated (far from any barriers) front with a flat bottom and found an unstable mode with a growth rate of the same magnitude as observed in the laboratory experiments. To model a coastal upwelling front, a coastal barrier and bottom topography are included in this study. Previous models show that active lower layers destabilize, while sloping bottom topography may stabilize the system. Since the coastal upwelling front environment contains both these features, it is of interest to investigate the net effect of the stability of the basic state flow.

The remainder of this paper is organized as follows. First, the particular model to be studied is described, and the governing equations of motion are stated. Next, conservation statements are derived and used to obtain general stability criteria. These criteria are used to determine a priori whether a particular frontal configuration is favorable for the growth of unstable disturbances. In a companion paper [Barth, this issue] (hereinafter referred to as part 2) the characteristics of unstable waves present for a variety of basic state flows and model geometries are presented. A comparison of the model calculations to observations from several upwelling regions is also presented in part 2.

2. MODEL DESCRIPTION

A simple, two-layer shallow water equation model with a rigid lid on an f plane is employed here. The stability analysis is carried out both for an inviscid fluid and with linearized bottom friction. The model explicitly leaves out the effects of wind stress and mixing. The applicability of a stability model without wind stress may be rationalized in two ways. First, coastal winds often become "upwelling-favorable" (blowing alongshore with the coast to the left in the northern hemisphere) for a period of a few days then relax or change direction [Huyer, 1983]. Therefore this instability model may be thought of as formally applying after one of these upwelling events. Second, the model may be appropriate even in the presence of a wind stress. In the real ocean, dissipation (e.g., via interfacial friction) will provide a sink of energy so that the wind-forced system may reach a steady state. If the dissipation is strong enough to effect this balance but weak enough to leave the structure of the unstable waves essentially unchanged, then the wind stress will only affect the stability analysis indirectly through its effect on the mean flow field. Since the wind forcing does not directly enter the stability calculation, an unforced model may be appropriate. However, as commented on further in part 2, section 4, time-dependence in the basic state flow

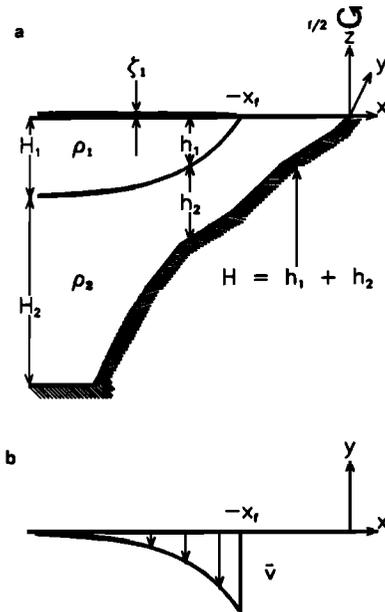


Fig. 2. (a) Model geometry (symbols are defined in text) and (b) upper layer basic state flow.

field as forced by a time-dependent wind stress (or for a steady wind stress before a steady state is established) may affect the stability properties of the system.

The model geometry is shown in Figure 2a. Two homogeneous layers of density ρ_1 and ρ_2 ($\rho_2 > \rho_1$) lie adjacent to a coastal barrier. The origin of the coordinate system is chosen to be at the coast with x , y , and z being the cross-front (positive onshore), alongfront, and vertical directions. The entire system is rotating about the z axis with an angular frequency $f/2$, where f is the Coriolis parameter. The layer thicknesses are denoted by h_1 and h_2 , while the bottom topography, which is an arbitrary function of x but assumed uniform in y , is given by $H = h_1 + h_2$. The sea surface elevation is denoted by ζ_1 . The surface front, modelled as the interface between the layers of different densities, lies parallel to the coast at the point x_f ($x_f < 0$), offshore of the coastal barrier. The sloping interface and bottom adjoin a distant flat-bottom region (representing the deep ocean offshore of the continental margin) with constant layer depths (H_1, H_2). A basic state alongfront flow (\bar{v}) which is uniform in y , independent of time (t) and in geostrophic balance exists in the upper layer (Figure 2b). For simplicity there is no basic state flow in the lower layer.

Before stating the governing equations, the field variables can be nondimensionalized as follows [e.g., Killworth *et al.* 1984]:

$$\begin{aligned} (x_*, y_*) &= R(x, y) \\ (u_{1*}, v_{1*}, u_{2*}, v_{2*}) &= (g'H_1)^{1/2}(u_1, v_1, u_2, v_2) \\ t_* &= f^{-1}t \\ (h_{1*}, h_{2*}) &= H_1(h_1, h_2) \\ \zeta_{1*} &= \delta H_1 \zeta_1 \end{aligned}$$

The variables subscripted with an asterisk are dimensional, and the numerical subscripts on the velocity components (u cross-front, v alongfront) indicate either the upper or lower layer. Horizontal length is scaled by the internal Rossby

radius of deformation, $R = (g'H_1)^{1/2}/f$, where the reduced acceleration due to gravity is given by $g' = g(\rho_2 - \rho_1)/\rho_2 = g\delta$. Note that the density defect, δ , is much less than 1. For typical coastal upwelling fronts $R \sim 5 - 10$ km. Velocities are nondimensionalized by the internal gravity wave phase speed, $(g'H_1)^{1/2}$, which is typically $50 - 100$ cm s^{-1} . Time is scaled by f^{-1} . Note that the sea surface elevation and layer depths are scaled such that their nondimensional versions are of the same order.

The nondimensional, two-layer, inviscid shallow water equations are, using subscripts to denote partial differentiation,

$$\frac{D_1 u_1}{Dt} - v_1 = -\zeta_{1x} \quad (1a)$$

$$\frac{D_1 v_1}{Dt} + u_1 = -\zeta_{1y} \quad (1b)$$

$$h_{1t} + [u_1(h_1 + \delta\zeta_1)]_x + [v_1(h_1 + \delta\zeta_1)]_y = 0 \quad (1c)$$

for the upper layer and

$$\frac{D_2 u_2}{Dt} - v_2 = -\zeta_{Tx} \quad (2a)$$

$$\frac{D_2 v_2}{Dt} + u_2 = -\zeta_{Ty} \quad (2b)$$

$$-h_{1t} + (u_2 h_2)_x + (v_2 h_2)_y = 0 \quad (2c)$$

for the lower layer where

$$\zeta_T = (1 - \delta)\zeta_1 - h_1$$

is essentially the lower layer "pressure." Here

$$\frac{D_i}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x} + v_i \frac{\partial}{\partial y} \quad i = 1, 2 \quad (3)$$

Linearized bottom friction, which has been demonstrated to be an important process in the coastal ocean [e.g., Allen, 1984], may be included in (2) (see the appendix in part 2).

The solution of (1)-(3) is simplified by invoking the geostrophic momentum approximation [Hoskins, 1975], which makes the governing equations linear in the eigenvalue (complex frequency; see the appendix in part 2). The approximation replaces the fluid velocities by their geostrophic values when acted upon by the substantial derivative (3). Specifically, the following substitutions are made:

$$\frac{D_i u_i}{Dt} = \frac{\partial u_{ig}}{\partial t} + u_i \frac{\partial u_{ig}}{\partial x} + v_i \frac{\partial u_{ig}}{\partial y} \quad i = 1, 2$$

$$\frac{D_i v_i}{Dt} = \frac{\partial v_{ig}}{\partial t} + u_i \frac{\partial v_{ig}}{\partial x} + v_i \frac{\partial v_{ig}}{\partial y}$$

where

$$\begin{aligned} u_{1g} &= -\zeta_{1y} \\ v_{1g} &= \zeta_{1x} \\ u_{2g} &= -\zeta_{Ty} \\ v_{2g} &= \zeta_{Tx} \end{aligned}$$

are the geostrophic layer velocities. It is important to retain the full (geostrophic and ageostrophic) advecting velocities in (3). The approximation is valid when the magnitude of

the time rate of change of the velocity vector is small compared with the magnitude of the Coriolis force [Hoskins, 1975], or, dimensionally,

$$\left| \frac{D\vec{v}_*}{Dt_*} \right| \ll |f\vec{v}_*|$$

Hoskins [1975] has demonstrated that the approximation works well in frontal regions with large horizontal shear provided the curvature vorticity (the turning of the flow along a streamline) is not large.

Another way to develop the geostrophic momentum approximation is to substitute for the velocities acted upon by the substantial derivative in (1a) and (1b) from the velocities in the Coriolis terms in (1b) and (1a) to obtain

$$v_1 = \zeta_{1x} - \frac{D_1\zeta_{1y}}{Dt} - \frac{D_1^2 v_1}{Dt^2} \tag{4a}$$

$$u_1 = -\zeta_{1y} - \frac{D_1\zeta_{1x}}{Dt} - \frac{D_1^2 u_1}{Dt^2} \tag{4b}$$

The lower layer equations (2) can be treated in a similar manner. The geostrophic momentum approximation is then obtained by neglecting the last term in each of the two equations. That is,

$$\frac{D_1^2 v_1}{Dt^2} \ll v_1$$

$$\frac{D_1^2 u_1}{Dt^2} \ll u_1$$

However, since most fronts are long in one direction and short in the other, the coordinate parallel to the surface front may be rescaled as $y_* = (R/\epsilon)y$, where $\epsilon \ll 1$. This small parameter can be identified as the alongfront wave number. With this scaling, (4) becomes

$$v_1 = \zeta_{1x} - \epsilon^2 \frac{D_1\zeta_{1y}}{Dt} - \epsilon^2 \frac{D_1^2 v_1}{Dt^2}$$

$$u_1 = -\zeta_{1y} - \frac{D_1\zeta_{1x}}{Dt} - \epsilon^2 \frac{D_1^2 u_1}{Dt^2}$$

At lowest order, the flow is geostrophic in the alongfront direction. The $O(1)$ equations have been called "semi-geostrophic" [Pedlosky, 1986]. In order to obtain results at higher wave numbers (shorter alongfront scales) it is necessary to include terms of $O(\epsilon^2)$. The geostrophic momentum approximation includes the term $D\zeta_{1y}/Dt$ in the alongfront momentum balance but arbitrarily neglects the final terms in each of the two equations. The geostrophic momentum approximation allows results to be found at high wave numbers, but may be suspect because of this arbitrary truncation of the $O(\epsilon^2)$ equations. The potential failure of the geostrophic momentum approximation for short alongfront perturbations has been noted by *McWilliams and Gent* [1980]. In this study, reliable results (as verified by direct comparison to the shallow water equation results of *Killworth et al.* [1984]) are found at all wave numbers for a flow with uniform basic state potential vorticity in the upper layer. As discussed further in part 2, section 2.1, the geostrophic momentum approximation fails to accurately predict the growth rate of high wave number unstable modes for some basic state flow configurations. However, it turns out that the properties of the fastest growing modes for all

the model fronts studied here are accurately predicted using the geostrophic momentum approximation. In any case, this approximation should be used with caution due to the arbitrary way in which it is "derived" from the shallow water equations.

From (4) under the geostrophic momentum approximation, the two-layer momentum equations are

$$\frac{D_1\zeta_{1y}}{Dt} + v_1 = \zeta_{1x} \tag{5a}$$

$$\frac{D_1\zeta_{1x}}{Dt} + u_1 = -\zeta_{1y} \tag{5b}$$

$$h_{1t} + [u_1(h_1 + \delta\zeta_1)]_x + [v_1(h_1 + \delta\zeta_1)]_y = 0 \tag{5c}$$

for the upper layer and

$$\frac{D_2\zeta_{2y}}{Dt} + v_2 = -\zeta_{2x} \tag{6a}$$

$$\frac{D_2\zeta_{2x}}{Dt} + u_2 = -\zeta_{2y} \tag{6b}$$

$$-h_{2t} + (u_2 h_2)_x + (v_2 h_2)_y = 0 \tag{6c}$$

for the lower layer where D_i/Dt ($i = 1, 2$) is again given by (3).

The nondimensional field variables are now expanded into a basic state (denoted by an overbar) and a perturbation (primed quantities). For the upper layer

$$\begin{aligned} h_1(x, y, t) &= (1 - \delta)\overline{h_1}(x) - \zeta'_2(x, y, t) \\ \zeta_1(x, y, t) &= \overline{\zeta_1}(x) + \zeta'_1(x, y, t) \\ u_1(x, y, t) &= \overline{u_1}(x) + u'_1(x, y, t) \\ v_1(x, y, t) &= \overline{v_1}(x) + v'_1(x, y, t) \end{aligned}$$

and for the lower layer

$$\begin{aligned} h_2(x, y, t) &= \overline{h_2}(x) + \zeta'_2(x, y, t) \\ \zeta_2(x, y, t) &= \overline{\zeta_2}(x) + \zeta'_T(x, y, t) \\ u_2(x, y, t) &= \overline{u_2}(x, y, t) \\ v_2(x, y, t) &= \overline{v_2}(x, y, t) \end{aligned}$$

The basic state flow only exists in the upper layer and is uniform alongshore, independent of time and in geostrophic balance

$$\overline{v}(x) = \overline{\zeta_1}_x(x) = \overline{h_1}_x(x) \tag{7}$$

Substituting these expressions into (5) and (6), linearizing about the basic state, and dropping primes, the geostrophic momentum equations for the perturbations become

$$\zeta_{1yt} + \overline{v}\zeta_{1yy} + v_1 = \zeta_{1x} \tag{8a}$$

$$\zeta_{1xt} + \overline{v}\zeta_{1xy} + (1 + \overline{v}_x)u_1 = -\zeta_{1y} \tag{8b}$$

$$-\zeta_{Tt} + (u_1\overline{h_1})_x + [v_1\overline{h_1} + \overline{v}(\zeta_1 - \zeta_T)]_y = 0 \tag{8c}$$

for the upper layer and

$$\zeta_{2yt} + v_2 = \zeta_{2x} \tag{9a}$$

$$\zeta_{2xt} + u_2 = -\zeta_{2y} \tag{9b}$$

$$\zeta_{Tt} + (u_2\overline{h_2})_x + (v_2\overline{h_2})_y = 0 \tag{9c}$$

for the lower layer where now

$$\zeta_T = (1 - \delta)\zeta_1 + \zeta_2 \tag{10}$$

Note that in (8c) terms of $O(\delta)$ have also been ignored. While these equations are eventually solved numerically (by a technique described in the appendix to part 2) for a variety of specific basic state flows (see part 2), it is useful to examine them first to see if any general statements about the stability of the system can be established.

3. CONSERVATION STATEMENTS AND A STABILITY THEOREM

While the existence of unstable waves on a particular basic state flow can be determined numerically, it is advantageous to have a general set of criteria for determining a priori whether that configuration is favorable for the growth of unstable disturbances. In this section, such criteria are developed through the use of global conservation statements for energy, momentum, and potential vorticity. These conservation statements are also useful in the dynamical interpretation of the instability mechanism, since they help to elucidate the details of energy transfer in the system. Differences between the conservation statements derived using the geostrophic momentum equations and the analogous quasi-geostrophic statements are noted. While the details of the geostrophic momentum formulation differ from the full shallow water development, all the essential differences between the latter and quasi-geostrophic theory are retained in the approximate set used here. Much of the present development parallels the work of *Hayashi and Young* [1987] (hereinafter referred to as HY) who studied a one-layer, reduced gravity, two-front model on an equatorial β plane.

3.1 Conservation Statements

The full nonlinear, geostrophic momentum equations (i.e., before expanding in a basic state and a perturbation), (5) and (6), lead to conservation statements for potential vorticity in the upper layer

$$\frac{D_1 q_1}{Dt} = 0 \tag{11a}$$

where

$$q_1 = \frac{1 + \zeta_{1xx} + \zeta_{1yy} - \zeta_{1xy}^2 + \zeta_{1xx}\zeta_{1yy}}{h_1} \tag{11b}$$

and in the lower layer

$$\frac{D_2 q_2}{Dt} = 0 \tag{12a}$$

where

$$q_2 = \frac{1 + \zeta_{2xx} + \zeta_{2yy} - \zeta_{2xy}^2 + \zeta_{2xx}\zeta_{2yy}}{h_2} \tag{12b}$$

(see the appendix for details). Again, D_i/Dt ($i = 1, 2$) is given by (3) and contains the full advecting velocities. These definitions of potential vorticity are identifiable with the traditional shallow water forms. One difference is that the advected relative vorticity has been replaced with its geostrophic value, consistent with the geostrophic momentum approximation. The extra terms in the numerators of (11b) and (12b) represent an ageostrophic component of the potential vorticity.

Similarly, an expression for the conservation of energy [Barth, 1987] is

$$\frac{\partial E}{\partial t} = 0 \tag{13a}$$

where

$$E = \frac{1}{2} \int \left[h_1 (\zeta_{1x}^2 + \zeta_{1y}^2) + h_2 (\zeta_{2x}^2 + \zeta_{2y}^2) + h_1^2 \right] da \tag{13b}$$

The integral is defined over the whole domain of the fluid in the x direction and over one wavelength in the y direction. Again, note the geostrophic form of the kinetic energies. Finally, an expression for absolute y momentum [Barth, 1987] is

$$\frac{\partial M}{\partial t} = 0 \tag{14a}$$

where

$$M = \int \left[h_1 (\zeta_{1x} + x) + h_2 (\zeta_{2x} + x) \right] da \tag{14b}$$

The terms in the integrand of (14b) proportional to x arise because the system is in a rotating reference frame.

The expressions for q_1 , q_2 , E , and M are expanded into a basic state and a perturbation as in the previous section. The potential vorticity becomes

$$q_1 = Q_1(x) + q'_1(x, y, t) \tag{15a}$$

where

$$Q_1(x) = \frac{1 + \bar{v}_x}{h_1} \tag{15b}$$

represents the basic state and

$$q'_1(x, y, t) = \frac{\zeta'_{1xx} + (1 + \bar{v}_x)\zeta'_{1yy} - Q_1(\zeta'_1 - \zeta'_T)}{h_1} \tag{15c}$$

is the perturbation potential vorticity. Q_1 has the same form as in the shallow water equations, while q'_1 has the additional term $\bar{v}_x \zeta'_{1yy} / h_1$ which represents part of the ageostrophic perturbation potential vorticity. For the lower layer

$$q_2 = Q_2(x) + q'_2(x, y, t) \tag{16a}$$

where

$$Q_2(x) = \frac{1}{h_2} \tag{16b}$$

$$q'_2(x, y, t) = \frac{\zeta'_{2xx} + \zeta'_{2yy} - Q_2 \zeta'_2}{h_2} \tag{16c}$$

Since there is no basic state flow in the lower layer, the basic state potential vorticity there is governed solely by the change in layer depth. The linearized forms of (11a) and (12a) for the perturbation potential vorticities are (dropping primes)

$$q_{1t} + \bar{v}q_{1y} + u_1 Q_{1x} = 0 \tag{17}$$

$$q_{2t} + u_2 Q_{2x} = 0 \tag{18}$$

The expansion of the energy conservation statement is slightly more complicated than that of potential vorticity due to the integration over the domain of the fluid. The undisturbed upper layer containing the mean current occupies the area from x_f to $-\infty$ in the cross-front direction

(see Figure 2). This can be denoted by $\int_{-\infty}^{x_f} (\) dA$ where the capital A represents the undisturbed area of the upper layer. A disturbance in the fluid moves the front to a position $x_f + \epsilon$ where $\epsilon \ll 1$ for small-amplitude, linear theory. This extra area now occupied by the disturbed upper layer can be denoted by $\int_{x_f}^{x_f + \epsilon} (\) da'$. Note that this area is an $O(\epsilon)$ quantity. Thus the area integral is expanded as

$$\int_{-\infty}^{x_f + \epsilon} (\) da = \int_{-\infty}^{x_f} (\) dA + \int_{x_f}^{x_f + \epsilon} (\) da'$$

The expansion of E can now proceed paying attention to the expansion of the area integral at the same time.

The full energy (13b) can be written as (dropping primes)

$$E = E_0 + E_1 + E_2 + \text{higher-order terms} \quad (19a)$$

where

$$E_0 = \frac{1}{2} \int (\overline{h_1 v^2} + \overline{h_1^2}) dA \quad (19b)$$

is the "basic state" energy,

$$E_1 = \int \left[\overline{h_1 v} \zeta_{1x} - \frac{1}{2} \overline{v^2} \zeta_2 - \overline{h_1} \zeta_2 \right] dA + \frac{1}{2} \int \overline{v^2} (\overline{h_1} - \zeta_2) da \quad (19c)$$

is the "mean" energy (really the energy associated with the wave-mean flow interaction) and

$$E_2 = \frac{1}{2} \int \left[\overline{h_1} (\zeta_{1x}^2 + \zeta_{1y}^2) - 2\overline{v} \zeta_{1x} \zeta_2 + \overline{h_2} (\zeta_{Tx}^2 + \zeta_{Ty}^2) + \zeta_2^2 \right] dA \quad (19d)$$

is the "wave" energy. Note that both E_1 and E_2 have $O(\epsilon^2)$ energy contributions. The $O(\epsilon^2)$ terms in E_1 can arise from both the second integral (because it spans an area $O(\epsilon)$ wide) and from the $O(\epsilon^2)$ parts of ζ_1 and ζ_2 in the first integral. Note that in the quasi-geostrophic limit, the second integral in (19c) vanishes because the front can not surface.

The definition of wave energy (E_2) contains the kinetic energy (in its geostrophic form) of the two layers and the potential energy due to the displacement of the interface. In addition, there is another term $-\overline{v} \zeta_{1x} \zeta_2$ which is not positive definite. This term represents the correlation between geostrophic alongfront perturbation velocity ($\zeta_{1x} = v_{1g}$) and perturbation interface displacement. For this term to be negative the correlation must be such that the disturbance increases the total upper layer thickness where it decreases the total alongfront speed and vice versa. This pattern is illustrated in Figure 3. This term is not present in the definition of wave energy in quasi-geostrophic theory because deviations of the interface from its basic state value are assumed small. In the quasi-geostrophic case, the wave energy is always positive definite.

Since E_0 does not change with time, (13a) can be written as

$$\frac{\partial (E_1 + E_2)}{\partial t} = 0 \quad (20)$$

where terms of $O(\epsilon^3)$ and higher have been neglected. This statement is true for both stable and unstable disturbances. The following discussion is restricted to the case of an unstable disturbance. If (20) is integrated in time from some

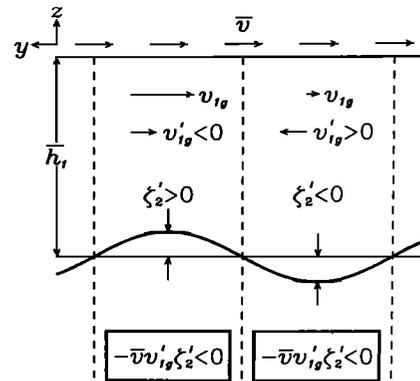


Fig. 3. Correlation between geostrophic alongfront perturbation velocity ($v'_{1g} = \zeta'_{1x}$) and perturbation interface displacement (ζ'_2) which results in a negative contribution (the boxed expression at the bottom of the figure) to the unstable wave energy (E_2) defined in (3.9d). Quantities are plotted as they appear when viewed from the two-layer side of the front. The basic state flow (\bar{v}) is toward $-y$ in an unperturbed layer of depth \bar{h}_1 . The sense of the correlation is such that the unstable wave increases the total alongfront geostrophic speed ($|v_{1g}| = |\bar{v} + v'_{1g}|$) where it decreases the total layer depth ($h_1 = \bar{h}_1 - \zeta'_2$) and vice versa.

initial state when there is no unstable disturbance present, then

$$E_1 + E_2 = 0$$

The disturbance energy is the energy in the disturbed system due to both the wave (E_2) and the modification of the mean flow (E_1). Following HY, this allows for three limiting possibilities of energy distribution in the disturbance between the wave itself and the mean flow modifications associated with it:

$$\begin{array}{ll} E_1 \rightarrow -\infty & E_2 \rightarrow +\infty \\ E_1 = 0 & E_2 = 0 \\ E_1 \rightarrow +\infty & E_2 \rightarrow -\infty \end{array}$$

Note that the energies going to infinity are in the context of small-amplitude linear theory, so that the true limiting values of the energies are scaled by ϵ , where $\epsilon \ll 1$. The first case represents the familiar, traditional instability process. As the unstable wave grows exponentially, its energy increases while that of the mean flow decreases. The second two cases arise because of the cross-term present in the definition of wave energy which allows for the possibility of "negative" energy. An unstable wave may grow in the system while its energy, as defined by (19d), and that of the mean flow, as defined by (19c), remain unchanged. HY term this "zero wave energy" instability. This case is relevant for the coastal model front when the basic state potential vorticity in the upper layer is uniform. In the last case, as the unstable wave grows, its energy becomes increasingly negative while that of the mean increases. This has been termed "negative wave energy" instability by HY.

These statements about the transfer of energy within a system containing an unstable disturbance are certainly counterintuitive. In quasi-geostrophic dynamics the basic state flow is always identified as a source of energy for the growing disturbance. Once the quasi-geostrophic approximation is abandoned, these other forms of instability are possible. HY thus suggest that the idea of instabilities re-

quiring a source of energy must be abandoned. These growing waves are possible because the total (mean plus wave) energy of the fluid is unaltered by the wave and the mean flow modifications associated with it.

The idea of negative energy instability is not new and is familiar in plasma physics [see *Cairns*, 1979]. *Cairns* [1979] has shown that for nonrotating stratified shear flows with step function velocity and density profiles, stable waves with negative energy can exist in the sense that exciting them lowers the total energy of the system. *Marinone and Ripa* [1984] found large-scale negative energy instabilities on an equatorial Gaussian jet in a one-layer, reduced gravity model. Zero energy instabilities have been studied by HY and also arise, though not commented on explicitly, in the work of *Griffiths et al.* [1982] and *Killworth et al.* [1984].

In the frontal studies of *Orlanski* [1968, 1969] only the positive definite wave energy

$$E_2^+ = \frac{1}{2} \int \left[\bar{h}_1 (\zeta_{1x}^2 + \zeta_{1y}^2) + \bar{h}_2 (\zeta_{2x}^2 + \zeta_{2y}^2) + \zeta_2^2 \right] dA \quad (21)$$

was considered in the energy balance requiring the definition of an "interaction kinetic energy" which obscured the interpretation of the energy transfers within the system as outlined here and by HY. The recent work on frontal models of *Griffiths et al.* [1982], *Killworth and Stern* [1982], *Killworth* [1983], and *Killworth et al.* [1984] did not thoroughly address the energetics of the unstable waves and, as in *Orlanski's* work, only concentrated on the positive definite part of the wave energy (equation (21)).

In a significant contribution to the understanding of the instability process in these counterintuitive cases, HY suggest that the unstable waves can be thought of as roughly a linear combination of resonating shear modes each of which would be stable if the other were not present. The two resonating waves must have opposite-signed disturbance (wave plus mean) energies so that the unstable mode has zero disturbance energy. The instability process is then an exchange of energy between the individual wave modes from the one with negative disturbance energy to the one with positive disturbance energy. In addition, destabilization by dissipation [*Holopainen*, 1961] can be understood in the same context because it provides a sink for a wave with negative disturbance energy [*Cairns*, 1979]. That is, the unstable mode grows as disturbance energy is removed from the mode and lost to dissipation. These ideas are commented on further in this section and confirmed in the unstable wave solutions of part 2, section 2.

The full y momentum can be expanded in an analogous way, yielding

$$M = M_0 + M_1 + M_2 + \text{higher-order terms} \quad (22a)$$

Here

$$M_0 = \int \left[\bar{h}_1 (\bar{v} + x) + \bar{h}_2 x \right] dA \quad (22b)$$

is the basic state momentum,

$$M_1 = \int \left[\bar{h}_1 \zeta_{1x} - \bar{v} \zeta_2 + \bar{h}_2 \zeta_{2x} \right] dA + \int \left[\bar{h}_1 (\bar{v} + x) + \bar{h}_2 x - \bar{v} \zeta_2 + \bar{h}_2 \zeta_{2x} \right] da \quad (22c)$$

is the momentum of the mean flow, and

$$M_2 = \int (\zeta_{Tx} - \zeta_{1x}) \zeta_2 dA \quad (22d)$$

is the wave momentum.

Expressions for the changes in wave energy (equation ((19d)) and momentum (equation (22d)) can also be obtained from the linearized perturbation equations (8) and (9). The result (derived in the appendix) for wave energy is

$$\begin{aligned} \frac{\partial E_2}{\partial t} &= - \int \left[\bar{h}_1 \bar{v}_x u_1 \zeta_{1x} + \bar{v}^2 u_1 \zeta_{1x} \right. \\ &\quad \left. + \bar{h}_1 \bar{v} (u_{1x} + v_{1y}) \zeta_{1x} - \bar{v} u_1 \zeta_2 - \bar{v} \bar{v}_x u_1 \zeta_2 \right] dA \\ &= - \frac{\partial E_1}{\partial t} \end{aligned} \quad (23)$$

The first two terms in the integrand represent horizontal Reynolds stresses in their geostrophic momentum form. The third term represents the vertical Reynolds stress, while the fourth term symbolizes the process of baroclinic instability. This latter process involves the flux of interface displacement in the cross-front direction or, in more physical terms, it is the movement of water in the cross-front direction in the two layers whose net effect is to flatten out the upwarded interface. As described by *Pedlosky* [1986] it is analogous to a slantwise form of convection or a downgradient flux of heat in a continuously stratified fluid. The baroclinic instability mechanism exchanges potential energy between the mean flow and the wave while the Reynolds stress terms exchange kinetic energy.

The final term in the integrand of (23) is not readily identifiable with a physical energy conversion process. It can, however, be related to the change in time of the displacement of the surface front [*Barth*, 1987]. This displacement can then be combined with the wave energy on the left-hand side of (23). The final result is

$$\begin{aligned} \frac{\partial}{\partial t} \left[E_2 - \frac{\bar{v}^3}{4} \int_{x=x_f} \epsilon^2 dy \right] &= - \int \left[\bar{h}_1 \bar{v}_x u_1 \zeta_{1x} \right. \\ &\quad \left. + \bar{v}^2 u_1 \zeta_{1x} + \bar{h}_1 \bar{v} (u_{1x} + v_{1y}) \zeta_{1x} - \bar{v} u_1 \zeta_2 \right] dA \end{aligned} \quad (24)$$

Since $\bar{v} < 0$, the deflection of the front is a positive definite addition to the wave energy. Now the change in time of wave energy and the displacement of the surface front due to the wave can be attributed completely to the Reynolds stresses acting on the basic state flow and the baroclinic conversion of potential energy.

The conservation of wave energy can also be written in terms of the cross-front eddy flux of perturbation potential vorticity. The details are contained in the appendix with the result

$$\frac{\partial E_2}{\partial t} = \int \bar{h}_1^{-2} \bar{v} u_1 q_1 dA + \frac{\partial}{\partial t} \int \bar{h}_1 \bar{v} \zeta_{1x} \zeta_{1y} dA \quad (25)$$

where E_2 is given by (19d) and q_1 by (15c). The final term on the right-hand side arises solely due to the geostrophic momentum approximation. While E_2 is the exact geostrophic form of the wave energy, the perturbation potential vorticity (q_1) contains both geostrophic and ageostrophic terms (see (15c)) so that the first integral on the right-hand side necessarily contains a purely ageostrophic component.

Therefore the final integral in (25) must also contain a purely ageostrophic component which can be identified as part of the ageostrophic wave energy missing in E_2 [Barth, 1987]. To get an expression relating the time rate of change of wave energy to the flux of perturbation potential vorticity alone (without the extra term in (25)), the full form of the energy (geostrophic and ageostrophic) must be included in E_2 . The resolution of this disparity between the shallow water form of the conservation of wave energy and the expression derived using the geostrophic momentum approximation is detailed by Barth [1987]. The final result is

$$\frac{\partial E_2^*}{\partial t} = \int \overline{h_1^{-2} \bar{v} u_1 q_1} dA \quad (26)$$

where the star denotes the full (geostrophic plus ageostrophic) wave energy defined by

$$E_2^* = \frac{1}{2} \int \left[\overline{h_1} (u_1^2 + v_1^2) - 2\bar{v} v_1 \zeta_2 + \overline{h_2} (u_2^2 + v_2^2) + \zeta_2^2 \right] dA$$

and q_1 remains given by (15c). The energy equation (23) can also be modified to express the conservation of total energy by including the Reynolds stresses due to the ageostrophic part of the velocity field.

The flux of perturbation potential vorticity in (26) can be related to the dispersion of particles within a basic state potential vorticity gradient. Using (17) and substituting

$$u_1 = \left(\frac{\partial}{\partial t} + \bar{v} \frac{\partial}{\partial y} \right) \eta_1 \quad (27)$$

where η_1 is the horizontal displacement of particles in the upper layer, the conservation of potential vorticity can be written as

$$\left(\frac{\partial}{\partial t} + \bar{v} \frac{\partial}{\partial y} \right) (q_1 + \eta_1 Q_{1x}) = 0 \quad (28)$$

If this expression is integrated with q_1 assumed initially to be zero, then

$$q_1 = -\eta_1 Q_{1x} \quad (29)$$

Substituting (27) and (29) into (26) yields

$$\frac{\partial E_2^*}{\partial t} = -\frac{\partial}{\partial t} \int \overline{h_1^{-2} \bar{v} Q_{1x} (\eta_1^2/2)} dA \quad (30)$$

Clearly, if the basic state potential vorticity is uniform ($Q_{1x} = 0$), then E_2^* does not change with time. Integrating (30) with respect to time gives E_2^* equal to a constant, and if the initial state contains no disturbance, then this constant must be identically zero. In quasi-geostrophic theory a uniform potential vorticity basic state does not satisfy the necessary conditions for instability, so E_2^* is always zero, which is in agreement with (30). By abandoning the restrictions required by quasi-geostrophic theory, it has been shown above that an unstable mode may exist if the total disturbance energy ($E_1^* + E_2^*$) is equal to zero. The existence of this "zero wave energy" mode of instability has been confirmed by Griffiths *et al.* [1982] and by HY and is demonstrated for the two-layer coastal upwelling front model in part 2, section 2. From (30) it is also clear that the sign of E_2^* can be determined from the sign of Q_{1x} . For $Q_{1x} < 0$ the wave energy is negative as in the study of Marinone and Ripa [1984]. The one-layer unstable mode of Killworth [1983] is also of this type. For $Q_{1x} > 0$ the traditional positive wave energy unstable mode is recovered.

It is important to note that an equation like (26) relating the change of wave energy to the flux of perturbation potential vorticity cannot be written for the positive definite quantity E_2^+ given by (21). Consequently, the time rate of change of E_2^+ cannot be related to the dispersion of particles in an unstable wave like in (30). This lack of connection between E_2^+ and particle displacements (a property of unstable waves whose increasing dispersion with time is a fundamental diagnostic of instability) rules out E_2^+ as an appropriate definition of wave energy. Instead, E_2 (or E_2^*) is a more useful measure of energy in a growing wave.

Conservation of wave y momentum derived from the linearized perturbation equations (8) and (9) (see the appendix for details) is

$$\begin{aligned} \frac{\partial M_2}{\partial t} = & \int \left[\overline{h_1^{-2} u_1 q_1} + \overline{h_2^{-2} u_2 q_2} \right] dA \\ & + \frac{\partial}{\partial t} \int \left[\overline{h_1 \zeta_{1x} \zeta_{1yy}} + \overline{h_2 \zeta_{2x} \zeta_{2yy}} \right] dA \end{aligned} \quad (31)$$

where the extra terms arise in an analogous manner to those in (25). They can be removed by considering the full y momentum with the result

$$\frac{\partial M_2^*}{\partial t} = \int \left[\overline{h_1^{-2} u_1 q_1} + \overline{h_2^{-2} u_2 q_2} \right] dA \quad (32)$$

where the full (geostrophic plus ageostrophic) wave momentum is defined by

$$M_2^* = \int (v_2 - v_1) \zeta_2 dA$$

Using (27), (29), and similar expressions for the lower layer, the conservation of wave y momentum can be related to the flux of perturbation potential vorticity in both layers

$$\frac{\partial M_2^*}{\partial t} = -\frac{\partial}{\partial t} \int \left[\overline{h_1^{-2} Q_{1x} (\eta_1^2/2)} + \overline{h_2^{-2} Q_{2x} (\eta_2^2/2)} \right] dA \quad (33)$$

3.2 Stability Theorem

Using the conservation statements for energy and y momentum, a general stability theorem can be derived. The method employed here is an extension of that of Ripa [1983] who developed a theorem for a one-layer, reduced gravity model on an equatorial β plane or a sphere. The definitions of E_2^* and M_2^* can be combined using an arbitrary constant γ

$$\begin{aligned} E_2^* - \gamma M_2^* = & \frac{1}{2} \int \left[\overline{h_1} (u_1^2 + v_1^2) + \overline{h_2} (u_2^2 + v_2^2) \right. \\ & \left. - 2(\bar{v} - \gamma) v_1 \zeta_2 + \zeta_2^2 - 2\gamma v_2 \zeta_2 \right] dA \end{aligned} \quad (34)$$

The integrand can be rewritten by completing the square of the terms involving v_1 and v_2 with the result

$$\begin{aligned} E_2^* - \gamma M_2^* = & \frac{1}{2} \int \left\{ \left[\overline{h_1}^{1/2} v_1 - \frac{(\bar{v} - \gamma) \zeta_2}{\overline{h_1}^{1/2}} \right]^2 \right. \\ & \left. + \left[\overline{h_2}^{1/2} v_2 - \frac{\gamma \zeta_2}{\overline{h_2}^{1/2}} \right]^2 + \overline{h_1} u_1^2 \right. \\ & \left. + \overline{h_2} u_2^2 + \zeta_2^2 - \frac{(\bar{v} - \gamma)^2 \zeta_2^2}{\overline{h_1}} - \frac{\zeta_2^2 \gamma^2}{\overline{h_2}} \right\} dA \end{aligned}$$

If the last three groups of terms are combined as

$$\zeta_2^2 \left[1 - \frac{(\bar{v} - \gamma)^2}{h_1} - \frac{\gamma^2}{h_2} \right] \tag{35}$$

the entire integrand is positive definite if

$$(\bar{v} - \gamma)^2 + \gamma^2 \mu \leq \bar{h}_1 \tag{36}$$

for all x where $\mu = \bar{h}_1/\bar{h}_2$ is the depth ratio. By setting the arbitrary constant γ equal to zero, (36) is satisfied if the magnitude of the upper layer mean flow is everywhere less than the internal gravity wave phase speed $((\bar{h}_1)^{1/2})$ nondimensionally. This type of flow is known as "subcritical." With a deep lower layer ($\mu \ll 1$), allowing γ to be nonzero allows \bar{v} to be supercritical somewhere in the flow and still satisfy (36). The inclusion of a finite depth lower layer makes (36) difficult to satisfy for the frontal flows of interest here.

The conservation statements for E_2^* and M_2^* , (30) and (33), can also be combined with the use of the same arbitrary constant γ to yield

$$\frac{\partial(E_2^* - \gamma M_2^*)}{\partial t} = -\frac{\partial}{\partial t} \int \left[\bar{h}_1^{-2} (\bar{v} - \gamma) Q_{1x} (\eta_1^2/2) - \bar{h}_2^{-2} \gamma Q_{2x} (\eta_2^2/2) \right] dA$$

Integrating with respect to time, this becomes

$$\underbrace{(E_2^* - \gamma M_2^*)}_{\text{const}} + \int \left[\bar{h}_1^{-2} (\bar{v} - \gamma) Q_{1x} (\eta_1^2/2) - \bar{h}_2^{-2} \gamma Q_{2x} (\eta_2^2/2) \right] dA = \text{const}$$

where the basic state potential vorticity gradients are obtained from (15b) and (16b). If each of the three groups of terms in this expression is positive, then no increase in wave properties (e.g., energy, particle dispersion) with time is allowed. A mixture of positive and negative terms can allow growth of the wave amplitude while still satisfying this expression. The first group are positive if (36) is satisfied as discussed above. Requiring the final two groups of terms to be positive leads to the statement of a general stability theorem:

If there exists any value of γ such that

$$(\bar{v} - \gamma)^2 + \gamma^2 \mu \leq \bar{h}_1 \tag{37a}$$

$$(\bar{v} - \gamma) Q_{1x} \geq 0 \tag{37b}$$

$$\gamma Q_{2x} \leq 0 \tag{37c}$$

for all x , then the flow is stable to infinitesimal perturbations.

These conditions are sufficient to insure stability, and (37) is essentially a two-layer version of Ripa's [1983] theorem. The extension to an arbitrary number of layers requires the addition of statements like (37c) for each layer. The stability criteria (37) are also similar to those of Long (unpublished manuscript, 1987) for continuously stratified, rotating flows. He finds a restriction on the vertical scale of the disturbance to which the flow is stable. This condition is analogous to (37a) if the scale of the disturbance is identified with the vertical wavelength of a long internal wave in a continuously stratified fluid. In the present study, the vertical scale of the disturbances is set by the layer depths. In either case, the flow is stable if the mean flow is everywhere less than

the internal gravity wave phase speed (and (37b) and (37c) are satisfied).

Quasi-geostrophic flows are generally weak (i.e., slow relative to the internal gravity wave speed), so they readily satisfy (37a). The remaining conditions (37b) and (37c) are just the familiar requirement that a change in the sign of the basic state potential vorticity gradient exist in order to get instability. Specifically, (37b) is just Fj\o rtoft's [1950] theorem (with γ equal to the value of \bar{v} at the inflection point) or equivalent to that of Rayleigh [1880] and Kuo [1949] (with γ outside the range of \bar{v}). The additional constraint (37c) allows the possibility for instability even if (37b) is satisfied by allowing the change in sign of the basic state potential vorticity gradient to occur between layers.

For the strong flows associated with frontal regions, unstable waves may still exist even if there is no change in the sign of the basic state potential vorticity gradient. This can occur if the first condition (37a) is violated as discussed above in association with (36). Recent reduced gravity models [Killworth and Stern, 1982; Griffiths et al. 1982; HY] have confirmed the existence of unstable modes in the absence of a change in sign of the basic state potential vorticity gradient. The existence of such a mode on the coastal upwelling front (as suggested by the analysis below) is verified in part 2, section 2.

The stability theorem (37) can now be applied to the coastal upwelling model by making an explicit choice for $\bar{h}_1(x)$ and via geostrophy (7), fixing the basic state upper layer flow. The choice for the interface profile (motivated by the results from models of coastal upwelling formation; see section 1) is a family of exponentials given by

$$\begin{aligned} \bar{h}_1(x) &= 1 - \exp[\alpha(x - x_f)] & x \leq x_f \\ \bar{h}_1(x) &= 0 & x > x_f \end{aligned} \tag{38a}$$

so that

$$\bar{v}(x) = -\alpha \exp[\alpha(x - x_f)] \quad x \leq x_f \tag{38b}$$

A uniform potential vorticity basic state has $\alpha = 1.0$. For $\alpha < 1.0$ the interface is less steeply sloping than the uniform potential vorticity front (e -folding length greater than 1.0, which in dimensional units is the internal Rossby radius of deformation), and for $\alpha > 1.0$ the front is more steeply sloping (e -folding length less than R).

For a uniform potential vorticity front ($\alpha = 1.0$), (37b) is automatically satisfied. Lower layer perturbation velocities vanish for a deep lower layer ($\mu \ll 1$) so that (37c) is automatically satisfied. For quasi-geostrophic flow, satisfying (37b) would alone be sufficient to insure stability. However, for these ageostrophic models, (37a) must also be satisfied. With (38) this condition becomes

$$[\exp(x - x_f) + \gamma]^2 \leq 1 - \exp(x - x_f)$$

which is satisfied for all x if $\gamma = -1$. A one-layer, reduced gravity model with uniform basic state potential vorticity is thus stable to infinitesimal perturbations of all scales. This result was also obtained by Paldor [1983], who used a Rayleigh integral technique applied directly to the governing equations.

The stability of nonuniform potential vorticity ($\alpha \neq 1.0$) flows with deep lower layers can also be determined. Substituting (38) into the quasi-geostrophic constraint (37b) yields

$$\{\alpha \exp[\alpha(x - x_f)] + \gamma\} Q_{1x} \leq 0$$

For $\alpha < 1$ ("shallow" interface profiles), the upper layer basic state potential vorticity gradient (Q_{1x}) is greater than zero and this expression is satisfied for all x if $\gamma \leq -\alpha$. For $\alpha > 1$ ("steep" interface profiles), $Q_{1x} \leq 0$ and the inequality holds for all x if $\gamma \geq 0$. Therefore quasi-geostrophic stability theory implies that these flows are stable. However, as in the case of the uniform potential vorticity flows discussed above, satisfaction of (37a) must also hold to insure stability. Substituting (38) into (37a) yields

$$\{\alpha \exp[\alpha(x - x_f)] + \gamma\}^2 \leq 1 - \exp[\alpha(x - x_f)]$$

This inequality is satisfied for $\gamma = -\alpha$, so that only flows with "shallow" interface profiles ($\alpha < 1.0$) are stable. This can be rationalized by realizing that $\alpha \ll 1.0$ is the correct limit to recover quasi-geostrophic flow (ignoring the $O(1)$ change in layer depth). For $\alpha > 1.0$, or "steep" interface profiles, the one-layer front may be unstable. This dependence on α of the stability of the front is in agreement with the results of Killworth [1983], who analytically solved the governing equations in a long wave limit by an involved boundary layer analysis.

The above discussion pertains to a model with a deep lower layer. With a finite depth lower layer and a flat bottom, the lower layer basic state potential vorticity gradient (Q_{2x}) is less than zero. For uniform potential vorticity in the upper layer ($\alpha = 1.0$) the inequalities (37b) and (37c) are satisfied if $\gamma \geq 0$. Thus quasi-geostrophic stability theory insures the stability of this flow. However, no choice of γ satisfies (37a), leading to the possibility of unstable modes. For nonuniform upper layer basic state potential vorticity the quasi-geostrophic constraints (37b) and (37c) imply stability for "steep" interface profiles ($\alpha > 1.0$; Q_{1x} and Q_{2x} are of one sign if $\gamma \geq 0$) and potential instability for "shallow" interface profiles ($\alpha < 1.0$). Again, the violation of the additional constraint (37a) implies the existence of unstable modes for any choice of interface profile. Finally, slopes in the bottom topography may exist which satisfy (37c). However, the inequality (37a) still does not hold for all x , giving the possibility for unstable modes.

In summary, the stability theorem derived here (equation (37)) suggests that coastal upwelling fronts (since they exist in shallow water) may be unstable no matter what the basic state flow configuration is. In violating sufficient conditions for stability the flow satisfies necessary conditions for instability. It is still essential to verify that unstable waves do exist on the coastal upwelling front, and this is done in part 2 by solving numerically (see the appendix in part 2 for details) the governing equations (8) and (9).

4. CONCLUSIONS

A simple two-layer shallow water equation model is used to investigate the stability of a coastal upwelling front over topography. Allowing divergent flow introduces a term in the definition of wave energy which is not positive definite. The presence of this term allows the growth of unstable disturbances with positive, zero, or negative wave energy. In the absence of external forcing, the total disturbance energy (the wave energy plus the change in the mean energy due to the presence of the unstable disturbance) must be zero. The growth of an unstable mode with positive wave energy is intuitive and is analogous to wave growth in a system governed by quasi-geostrophy. The nonintuitive idea of the growth of

a wave with zero or negative energy is rationalized in terms of the exchange of disturbance energies between two stable modes whose alliance results in the unstable wave.

By combining conservation statements for the global properties of the system (potential vorticity, energy, momentum), a stability theorem is established which allows the a priori determination of the stability of a coastal upwelling front. The theorem differs from the traditional quasi-geostrophic theorem by including an additional constraint on the basic state flow in order to insure stability. This additional constraint can be attributed directly to the presence of the term which is not positive definite in the definition of wave energy. The stability theorem suggests that coastal upwelling fronts are potentially unstable no matter what the basic state flow configuration is. This clearly illustrates the danger in applying stability criteria derived from quasi-geostrophic theory to frontal configurations.

The model presented here has been simplified by ignoring the effects of, for example, stratification underlying the surface front, interfacial friction, wind stress, and alongshore variations in the bottom topography or coastline configuration. These processes may affect the stability of a realistic coastal upwelling front (see the discussion in part 2, section 4.1). However, as detailed in part 2, reasonable agreement between the model-predicted properties of the fastest growing waves and observed scales of alongfront variability is found. This suggests that the simplified model captures the basic instability process.

APPENDIX

A1. Conservation of Potential Vorticity

The derivation of the conservation of potential vorticity in each layer from the full, nonlinear geostrophic momentum equations (5) and (6) is as follows. Note the differences from the traditional shallow water equation development [Pedlosky, 1986]. Details of the derivation for the upper layer are presented with the lower layer derivation following in an analogous manner. Taking the curl of the upper layer momentum equations [$\partial(5a)/\partial y + \partial(5b)/\partial x$] yields

$$\begin{aligned} \frac{D_1}{Dt}(\zeta_{1xx} + \zeta_{1yy}) + (u_{1x} + v_{1y}) + (v_{1x} + u_{1y})\zeta_{1xy} \\ + u_{1x}\zeta_{1xx} + v_{1y}\zeta_{1yy} = 0 \end{aligned} \quad (A1)$$

Rewriting (5c) as

$$(u_{1x} + v_{1y}) = -\frac{1}{(h_1 + \delta\zeta_1)} \frac{D_1}{Dt}(h_1 + \delta\zeta_1) \quad (A2)$$

allows $(u_{1x} + v_{1y})$ to be replaced in (A1) to yield

$$\begin{aligned} \frac{D_1}{Dt}(\zeta_{1xx} + \zeta_{1yy}) - \frac{1}{(h_1 + \delta\zeta_1)} \frac{D_1}{Dt}(h_1 + \delta\zeta_1) \\ + (v_{1x} + u_{1y})\zeta_{1xy} + u_{1x}\zeta_{1xx} + v_{1y}\zeta_{1yy} = 0 \end{aligned} \quad (A3)$$

To replace the term in (A3) involving $(v_{1x} + u_{1y})$, take $\partial(5a)/\partial x$ and add to $\partial(5b)/\partial y$ to obtain

$$\begin{aligned} 2\frac{D_1}{Dt}\zeta_{1xy} + (v_{1x} + u_{1y}) + (u_{1x} + v_{1y})\zeta_{1xy} \\ + (v_{1x} + 1)\zeta_{1yy} + (u_{1y} - 1)\zeta_{1xx} = 0 \end{aligned}$$

Substitute for $(u_{1x} + v_{1y})$ from (A2) and rearrange to get

$$(v_{1x} + u_{1y}) = -2\frac{D_1}{Dt}\zeta_{1xy} + \frac{\zeta_{1xy}}{(h_1 + \delta\zeta_1)}\frac{D_1}{Dt}(h_1 + \delta\zeta_1) - (v_{1x} + 1)\zeta_{1yy} - (u_{1y} - 1)\zeta_{1xx} \quad (A4)$$

Multiplying (A4) by ζ_{1xy} and substituting in (A3) yields

$$\begin{aligned} \frac{D_1}{Dt}(\zeta_{1xx} + \zeta_{1yy} - \zeta_{1xy}^2) + \frac{(\zeta_{1xy}^2 - 1)}{(h_1 + \delta\zeta_1)}\frac{D_1}{Dt}(h_1 + \delta\zeta_1) \\ + \zeta_{1xx}[u_{1x} - \zeta_{1xy}(u_{1y} - 1)] \\ + \zeta_{1yy}[v_{1y} - \zeta_{1xy}(v_{1x} + 1)] = 0 \end{aligned} \quad (A5)$$

Finally, the last two groups of terms in square brackets are replaced using ζ_{1xx} times (5a) plus ζ_{1yy} times (5b), or

$$\begin{aligned} \frac{D_1}{Dt}(\zeta_{1xx}\zeta_{1yy}) + \zeta_{1xx}[v_{1y} + (u_{1y} - 1)\zeta_{1xy} + v_{1y}\zeta_{1yy}] \\ + \zeta_{1yy}[u_{1x} + (v_{1x} + 1)\zeta_{1xy} + u_{1x}\zeta_{1xx}] = 0 \end{aligned}$$

As before, (A2) is used to replace terms proportional to $(u_{1x} + v_{1y})$ with the result

$$\begin{aligned} \frac{D_1}{Dt}(\zeta_{1xx}\zeta_{1yy}) \\ - (\zeta_{1xx}\zeta_{1yy} + \zeta_{1xx} + \zeta_{1yy})\frac{1}{(h_1 + \delta\zeta_1)}\frac{D_1}{Dt}(h_1 + \delta\zeta_1) \\ - \zeta_{1xx}[u_{1x} - \zeta_{1xy}(u_{1y} - 1)] \\ - \zeta_{1yy}[v_{1y} - \zeta_{1xy}(v_{1x} + 1)] = 0 \end{aligned} \quad (A6)$$

Adding (A5) and (A6) yields the final result from which (11a) and (11b) follow.

A2. Conservation of Wave Energy

An expression for the time rate of change of the wave energy E_2 (defined in (19d)) can be obtained directly from the linearized perturbation equations (8) and (9). Multiplying (8a) by $\overline{h_1}\zeta_{1y}$ and (8b) by $\overline{h_1}\zeta_{1x}$ and then adding yields

$$\begin{aligned} \frac{\partial}{\partial t} \left[\overline{h_1} \left(\frac{\zeta_{1x}^2 + \zeta_{1y}^2}{2} \right) \right] + \frac{\partial}{\partial x} \left[\overline{h_1}\overline{v} \left(\frac{\zeta_{1x}^2 + \zeta_{1y}^2}{2} \right) \right] \\ = -\overline{h_1}(1 + \overline{v_x})u_1\zeta_{1x} - \overline{h_1}v_1\zeta_{1y} \end{aligned} \quad (A7)$$

A similar operation on the lower layer equations (9) yields

$$\frac{\partial}{\partial t} \left[\overline{h_2} \left(\frac{\zeta_{Tx}^2 + \zeta_{Ty}^2}{2} \right) \right] = -\overline{h_2}(u_2\zeta_{Tx} + v_2\zeta_{Ty}) \quad (A8)$$

To include the potential energy due to the displacement of the interface, multiply (8c) by ζ_1 and (9c) by ζ_T and then add to get

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\zeta_2^2}{2} \right) + \zeta_1(u_1\overline{h_1})_x + \zeta_1[v_1\overline{h_1} - \overline{v}\zeta_2]_y \\ + \zeta_T(u_2\overline{h_2})_x + \zeta_T(v_2\overline{h_2})_y = 0 \end{aligned} \quad (A9)$$

where terms of $O(\delta)$ have been ignored. Adding (A7)–(A9) yields

$$\begin{aligned} \frac{\partial}{\partial t} \left[\overline{h_1} \left(\frac{\zeta_{1x}^2 + \zeta_{1y}^2}{2} \right) + \overline{h_2} \left(\frac{\zeta_{Tx}^2 + \zeta_{Ty}^2}{2} \right) + \frac{\zeta_2^2}{2} \right] \\ + \frac{\partial}{\partial x} (\overline{h_1}u_1\zeta_1 + \overline{h_2}u_2\zeta_T) \end{aligned}$$

$$\begin{aligned} + \frac{\partial}{\partial y} \left[\overline{h_1}\overline{v} \left(\frac{\zeta_{1x}^2 + \zeta_{1y}^2}{2} \right) + \overline{h_1}v_1\zeta_1 + \overline{h_2}v_2\zeta_T \right] \\ = -\overline{v_x}\overline{h_1}u_1\zeta_{1x} + \zeta_1(\overline{v}\zeta_2)_y \end{aligned} \quad (A10)$$

The cross term appearing in the definition of wave energy (equation (19d)) is obtained by multiplying (8b) by $-\overline{v}\zeta_2$ and (8c) by $\overline{v}\zeta_{1x}$ and then adding. The result is

$$\begin{aligned} \frac{\partial}{\partial t} (-\overline{v}\zeta_2\zeta_{1x}) - (1 + \overline{v_x})\overline{h_1}\overline{v}u_1\zeta_2 + \overline{v}\zeta_{1x}(u_1\overline{h_1})_x \\ + \overline{v}\overline{h_1}\zeta_{1x}v_{1y} - \overline{v}^2(\zeta_{1x}\zeta_2)_y - \overline{v}\zeta_2\zeta_{1y} = 0 \end{aligned} \quad (A11)$$

Adding (A11) to (A10) yields

$$\begin{aligned} \frac{\partial}{\partial t} \left[\overline{h_1} \left(\frac{\zeta_{1x}^2 + \zeta_{1y}^2}{2} \right) - \overline{v}\zeta_2\zeta_{1x} + \overline{h_2} \left(\frac{\zeta_{Tx}^2 + \zeta_{Ty}^2}{2} \right) + \frac{\zeta_2^2}{2} \right] \\ + \frac{\partial}{\partial x} (\overline{h_1}u_1\zeta_1 + \overline{h_2}u_2\zeta_T) \\ + \frac{\partial}{\partial y} \left[\overline{h_1}\overline{v} \left(\frac{\zeta_{1x}^2 + \zeta_{1y}^2}{2} \right) \right. \\ \left. + \overline{h_1}v_1\zeta_1 + \overline{h_2}v_2\zeta_T - \overline{v}^2\zeta_{1x}\zeta_2 - \overline{v}\zeta_2\zeta_1 \right] \\ = -\overline{v_x}\overline{h_1}u_1\zeta_{1x} + (1 + \overline{v_x})\overline{h_1}\overline{v}u_1\zeta_2 \\ - \overline{v}\zeta_{1x}(u_1\overline{h_1})_x - \overline{v}\overline{h_1}\zeta_{1x}v_{1y} \end{aligned}$$

Integrating over the domain of the fluid and applying the boundary conditions in the cross-front direction (basically, the cross-front flow in each layer vanishes far from the surface front or, in the case of the lower layer, vanishes because of a coastal barrier; see the appendix in part 2 for details) leads to equation (23).

A3. Derivation of $\partial E_2/\partial t \propto u_1q_1$

To relate $\partial E_2/\partial t$ to the cross-front eddy flux of perturbation potential vorticity, the Reynolds stress terms and baroclinic energy conversion term on the right-hand side of (23) must be rewritten:

$$\begin{aligned} \frac{\partial E_2}{\partial t} = \int \left[-(\overline{h_1}\overline{v_x}u_1\zeta_{1x} + \overline{v}^2u_1\zeta_{1x} + \overline{h_1}\overline{v}u_{1x}\zeta_{1x}) \right. \\ \left. + \overline{h_1}\overline{v}u_1Q_1\zeta_2 - \overline{h_1}\overline{v}v_{1y}\zeta_{1x} \right] dA \end{aligned} \quad (A12)$$

The term in parentheses can be rewritten as $\overline{h_1}\overline{v}u_1\zeta_{1xx}$ because $\partial(\overline{h_1}\overline{v}u_1\zeta_{1x})/\partial x$ vanishes over the interval. To rewrite the final term in the integrand of (A12), multiply (8a) by ζ_{1xy} and (8b) by ζ_{1yy} , then add to obtain

$$\begin{aligned} \zeta_{1xy}\zeta_{1yt} - \zeta_{1yy}\zeta_{1xt} + v_1\zeta_{1xy} - (1 + \overline{v_x})u_1\zeta_{1yy} \\ = \frac{\partial}{\partial y} \left(\frac{\zeta_{1x}^2 + \zeta_{1y}^2}{2} \right) \end{aligned}$$

Multiplying by $\overline{h_1}\overline{v}$ and then averaging over the domain of the fluid yields

$$\int \overline{h_1}\overline{v} \left[\zeta_{1xy}\zeta_{1yt} - \zeta_{1yy}\zeta_{1xt} + v_1\zeta_{1xy} - (1 + \overline{v_x})u_1\zeta_{1yy} \right] dA = 0 \quad (A13)$$

The first two terms may be replaced by

$$\begin{aligned} \int \bar{h}_1 \bar{v} \zeta_{1xy} \zeta_{1yt} dA &= - \int \bar{h}_1 \bar{v} \zeta_{1x} \zeta_{1yyt} dA, \\ - \int \bar{h}_1 \bar{v} \zeta_{1yy} \zeta_{1xt} dA &= \int \bar{h}_1 \bar{v} \left[-(\zeta_{1yy} \zeta_{1x})_t \right. \\ &\quad \left. + \zeta_{1x} \zeta_{1yyt} \right] dA \end{aligned}$$

so (A13) becomes

$$\int \bar{h}_1 \bar{v} \left[-(\zeta_{1yy} \zeta_{1x})_t + v_1 \zeta_{1xy} - (1 + \bar{v}_x) u_1 \zeta_{1yy} \right] dA = 0$$

Rearranging gives the desired result

$$- \int \bar{h}_1 \bar{v} v_1 \zeta_{1x} dA = \int \bar{h}_1 \bar{v} \left[(1 + \bar{v}_x) u_1 \zeta_{1yy} + (\zeta_{1yy} \zeta_{1x})_t \right] dA$$

Substituting into (A12) yields

$$\begin{aligned} \frac{\partial E_2}{\partial t} &= \int \bar{h}_1^{-2} \bar{v} u_1 \left[\frac{\zeta_{1xx} + (1 + \bar{v}_x) \zeta_{1yy} + Q_1 \zeta_2}{\bar{h}_1} \right] dA \\ &\quad + \frac{\partial}{\partial t} \int \bar{h}_1 \bar{v} \zeta_{1x} \zeta_{1yy} dA \end{aligned}$$

which is just (25), since q_1 is given by (15c). (Note that a term of $O(\delta)$ in the definition of q_1 has been ignored.)

A4. Conservation of Wave Momentum

Conservation of wave y momentum can be derived from the linearized perturbation equations (8) and (9) as follows. Multiplying (8b) by $-\zeta_2$ and (8c) by ζ_{1x} and adding yields

$$\begin{aligned} -(\zeta_2 \zeta_{1x})_t - (\zeta_2 \bar{v} \zeta_{1x})_y - (1 + \bar{v}_x) u_1 \zeta_2 \\ + (u_1 \bar{h}_1)_x \zeta_{1x} + \bar{h}_1 v_1 \zeta_{1x} = \zeta_{1y} \zeta_2 \end{aligned} \quad (A14)$$

A similar operation for the lower layer [(9b) ζ_2 + (9c) ζ_{Tx}] gives

$$\begin{aligned} (\zeta_2 \zeta_{Tx})_t + u_2 \zeta_2 + (u_2 \bar{h}_2)_x \zeta_{Tx} + \\ + \bar{h}_2 v_2 \zeta_{Tx} = -\zeta_2 \zeta_{1y} - \left(\frac{\zeta_2^2}{2} \right)_y \end{aligned} \quad (A15)$$

Adding (A14) to (A15) and averaging over the domain of the fluid yields

$$\begin{aligned} \frac{\partial}{\partial t} \int (\zeta_{Tx} - \zeta_{1x}) \zeta_2 dA &= \int \left[(1 + \bar{v}_x) u_1 \zeta_2 - (u_1 \bar{h}_1)_x \zeta_{1x} \right. \\ &\quad \left. - \bar{h}_1 v_1 \zeta_{1x} - u_2 \zeta_2 \right. \\ &\quad \left. - (u_2 \bar{h}_2)_x \zeta_{Tx} - \bar{h}_2 v_2 \zeta_{Tx} \right] dA \end{aligned}$$

The right-hand side is rewritten using manipulations similar to those used in the derivation of the conservation of energy with the result

$$\begin{aligned} \frac{\partial M_2}{\partial t} &= \int \bar{h}_1^{-2} u_1 \left[\frac{\zeta_{1xx} + (1 + \bar{v}_x) \zeta_{1yy} + Q_1 \zeta_2}{\bar{h}_1} \right] dA \\ &\quad + \frac{\partial}{\partial t} \int \bar{h}_1 \zeta_{1x} \zeta_{1yy} dA \\ &\quad + \int \bar{h}_2^{-2} u_2 \left[\frac{\zeta_{Tx} + \zeta_{Ty} - Q_2 \zeta_2}{\bar{h}_2} \right] dA \\ &\quad + \frac{\partial}{\partial t} \int \bar{h}_2 \zeta_{Tx} \zeta_{Ty} dA \end{aligned} \quad (A16)$$

where Q_1 and Q_2 are defined by (15b) and (16b). The quantities in square brackets are just the perturbation potential vorticities in each layer given by (15c) and (16c), so that (A16) becomes (31).

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